2 Prove by induction that, for all $N \geqslant 1$,

$$\sum_{n=1}^{N} \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(N+1)2^N}.$$ [5]

3 Let $v_1, v_2, v_3, \ldots$ be a sequence and let

$$u_n = nv_n - (n+1)v_{n+1},$$

for $n = 1, 2, 3, \ldots$. Find $\sum_{n=1}^{N} u_n$. [2]

In each of the following cases determine whether the series $u_1 + u_2 + u_3 + \ldots$ is convergent, and justify your conclusion. Give the sum to infinity where this exists.

(i) $v_n = n^{-\frac{1}{2}}$. [2]

(ii) $v_n = n^{-\frac{3}{2}}$. [2]

9231_s03_er

Question 2

About a half of all candidates answered this question without error. There were some notational confusions of N with n. A more serious error was the confusion of the inductive hypothesis with the result. The centre part of the argument is to show that $H_k \Rightarrow H_{k+1}$ for any integer positive integer k(*A*) and, as such, does not prove that H_k is true. The proof is completed by showing that H_1 is true (*B*). Of course, the stages (*A*) and (*B*) may be effected in either order. In the majority of responses the working at stage (*A*) was complete and accurate. Nevertheless, common errors here were the incorrect formation of term $k + 1$ and omission of essential detail in the subsequent working. At stage (*B*) which, evidently, some candidates thought was not worth bothering about, there was again lack of attention to detail.

Question 3

Most candidates appreciated that the evaluation of $S_N = \sum_{n=1}^{N} u_n$ could be effected by application of the difference method. Common incorrect answers were $v - (N + 1)v_{N+1}$ and $v_1 - (N + 2)v_{N+2}$.

(i) Only a minority wrote $S_N = 1 - (N + 1)^{\frac{1}{2}}$ before attempting to investigate the convergence (or otherwise) of $\sum_{n=1}^{\infty} u_n$. In this respect, an argument such as the following was expected. '$S_n = 1 - (N + 1)^{\frac{1}{2}} \to -\infty$ as $N \to +\infty$. Hence the infinite series $\sum_{n=1}^{\infty} u_n$ is not convergent.'

(ii) Likewise here, the corresponding argument should be as follows. '$S_N = 1 - (N+1)^{-\frac{1}{2}} \to 1 - 0 = 1$ as $N \to +\infty$'. Hence $\sum_{1}^{\infty} u_n$ is convergent and its sum to infinity is '1'. However, only a minority argued in this way. Thus the concept of convergence of an infinite series appears, overall, to have been poorly understood.

2 Given that

$$u_n = \frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1},$$

find $S_N = \sum_{n=N+1}^{2N} u_n$ in terms of N. [3]

Find a number M such that $S_N < 10^{-20}$ for all $N > M$. [3]

9231_w03_er

Question 2

There were very few complete and correct responses to this question.

Almost all candidates understood that an application of the difference method was required and went on to obtain a correct result for S_N in terms of N. However, some of the working was confused in that it was unclear what f(n) actually is when writing u_n = f(n – 1) – f(n).

In contrast, the second part of this question proved to be the major stumbling block of this paper and only a minority of the candidature made any significant progress here. This general failure was due, in almost all cases, to a supposition that the question, in effect, stated that there was only one possible value of M and that it was up to the candidate to find it. This misconception motivated many candidates to set their result for S_N, usually correct, equal to 10^{-20} and so to become involved in algebraically unrealistic objectives. However, it almost obvious that $S^N < \frac{1}{N^2}$, so that a suitable value of M can easily be obtained.

Answer: $M = 10^{10}$.

1 Use the relevant standard results in the List of Formulae to prove that

$$S_N = \sum_{n=1}^{N} (8n^3 - 6n^2) = N(N+1)(2N^2 - 1).$$ [2]

Hence show that

$$\sum_{n=N+1}^{2N} (8n^3 - 6n^2)$$

can be expressed in the form

$$N(aN^3 + bN^2 + cN + d),$$

where the constants a, b, c, d are to be determined. [2]

9231_s04_er

Question 1

Generally, this introductory question was answered accurately. Some of the methods adopted were sub-optimal with respect to use of examination time.

Most candidates began by writing $S_N = 8\left(\frac{1}{4}\right)N^2(N+1)^2 - 6\left(\frac{1}{6}\right) N(N+1)(2N+1)$, and then showed about the right amount of working to establish the required result. There were a few attempts to use induction, but whether correct, or not, these could not be awarded any credit, for the question specifically demands use of standard results in the List of Formulae.

In the second part of the question, not all candidates recognised that the required sum is $S_{2N} - S_N$ but instead worked from incorrect forms such as $S_{2N} - S_{N+1}$, or $S_{2N} - S_{N-1}$.

Moreover the correct $\sum_{n=N+1}^{2N}(8n^3 - 6n^2) = 2N(2N+1)(8N^2-1) - N(N+1)(2N^2-1)$ was not always accurately transformed to the displayed result.

Answer: $N(30N^3 + 14N^2 - 3N - 1)$.

5 Let

$$S_N = \sum_{n=1}^{N} (-1)^{n-1} n^3.$$

Find S_{2N} in terms of N, simplifying your answer as far as possible. [4]

Hence write down an expression for S_{2N+1} and find the limit, as $N \to \infty$, of $\frac{S_{2N+1}}{N^3}$. [3]

9231_w04_er

Question 5

In contrast to the earlier questions, most candidates found this question to be difficult so that complete answers were very much in a minority.

Some responses began with a decomposition of the form $S_{2N} = \sum_{n=1}^{2N} n^3 - k\sum_{n=1}^{N} n^3$ and this was followed by sensible attempts to sum the two series involved by means of the standard result $\sum_{n=1}^{N} n^3 = \frac{1}{4}n^2(n+1)^2$. However, more often than not, the summation limits were incorrect in at least one part of the decomposition and/or the implied value of k was wrong. Even less successful was the strategy of writing $S_{2N} = \sum_{n=1}^{N} \left[(2n-1)^3 - (2n)^3\right]$ (*) followed by separate attempts to sum each series. A few candidates, however, did see that (*) is equivalent to $\sum_{n=1}^{N} \left(-12n^2 + 6n - 1\right)$ and hence that $S_{2N} = -2N(N+1)(2N+1) + 3N(N+1) - N$, etc.

For the second part of the question, many innovative but incorrect results for S_{2N+1} were written down. Only about half of all candidates appeared to understand that $S_{2N+1} = S_{2N} + (2N+1)^3$ and only about half of these went on to produce a result such as $\frac{S_{2N+1}}{N^3} = \left(2 + \frac{1}{N}\right)^3 - \left(4 + \frac{3}{N}\right)$ from which the required limit can be obtained immediately.

Answer: $S_{2N} = -N^2(4N+3)$; $\lim_{N\to\infty}\left(\frac{S_{2N+1}}{N^3}\right) = 4$.

8 The sequence of real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^{\lambda},$$

where λ is a constant greater than 1. Prove by mathematical induction that, for $n \geqslant 2$,

$$a_n \geqslant 2^{g(n)},$$

where $g(n) = \lambda^{n-1}$. [6]

Prove also that, for $n \geqslant 2$, $\frac{a_{n+1}}{a_n} > 2^{(\lambda-1)g(n)}$. [3]

9231_w04_er

Question 8

By a long way, this was the least well answered question of the paper. In fact, a typical response did no more than set out the inductive hypothesis, H_k, and to verify it to be correct for $k = 2$, and then to go on to make no further progress in either part of the question.

In part **(i)** it was first required that H_k be defined as $a_k > 2^{g(k)}$ for some k and that subsequently it would be clearly stated that as $\frac{1}{a_k} > 0$, then $a_{k+1} > (a_k)^{\lambda}$, and hence that $H_k \Rightarrow a_{k+1} > 2^{\lambda g(k)} = 2^{\left(\lambda^k\right)} = 2^{g(k+1)}$, so that $H_k \Rightarrow H_{k+1}$.

The inductive argument would then be completed by stating that H_2 is true, since $a_2 = 2^{\lambda} = 2^{g(2)}$.

Fundamentally erroneous statements of the form $a_k + \frac{1}{a_k} \geqslant 2^{g(k)} + \frac{1}{2^{g(k)}}$ were very prevalent as also were arguments based on the binomial expansion of $\left(a_k + \frac{1}{a_k}\right)^{\lambda}$ as if λ is an integer. Use of the binomial series for a non-integer $\lambda > 1$ would, in the first instance, involve an infinite series which includes some negative terms. The working of such an argument into a rigorous form would be time consuming.

The proof of the displayed result in the second part of this question requires no more than a simple argument such as: $\frac{a_{n+1}}{a_n} = a_n^{\lambda-1}\left(1 + \frac{1}{a_n^2}\right)^{\lambda} > a_n^{\lambda-1} > \left[2^{g(n)}\right]^{\lambda-1} = 2^{(\lambda-1)g(n)}$.

However, this response was produced by only a small minority of candidates, whereas the erroneous argument $\left[a_n > 2^{g(n)} \text{ and } a_{n+1} > 2^{g(n+1)}\right] \Rightarrow \frac{a_{n+1}}{a_n} > \frac{2^{g(n+1)}}{2^{g(n)}}$ appeared in some form in more than half of all scripts.

1 Use the method of differences to find S_N, where

$$S_N = \sum_{n=N}^{N^2} \frac{1}{n(n+1)}. \qquad [3]$$

Deduce the value of $\lim_{N\to\infty} S_N$. [1]

9231_s05_er

Question 1

Almost all candidates produced some good work in response to this question. Common errors were the writing of S_N as $\sum_{n=1}^{N^2} f(n) - \sum_{n=1}^{N} f(n)$, where $f(n) = \frac{1}{n(n+1)}$, or even simply as $\sum_{n=1}^{N} f(n)$. However, the majority did work from $\sum_{n=1}^{N^2} f(n) - \sum_{n=1}^{N-1} f(n)$ to obtain the required sum function in terms of N.

The concept of a limit in this context appeared to be well understood by most candidates and the working here was generally accurate and complete.

Answers: $S_N = \frac{1}{N} - \frac{1}{N^2+1}$; $\lim_{N\to\infty} S_N = 0$.

2 The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and

$$u_{n+1} = -1 + \sqrt{(u_n + 7)}.$$

(i) Prove by induction that $u_n < 2$ for all $n \geqslant 1$. [4]

(ii) Show that if $u_n = 2 - \varepsilon$, where ε is small, then

$$u_{n+1} \approx 2 - \tfrac{1}{6}\varepsilon.$$ [2]

9231_w05_er

Question 2

(i) The majority of responses began with something like 'Let H_k be the inductive hypothesis $u_k < 2$ for all positive integers k'. Such a statement is clearly meaningless and shows a fundamental lack of understanding of mathematical induction. Instead, the argument should begin with the equivalent of 'Let H_k be the inductive hypothesis $u_k < 2$ for some positive integer k'.

The proof that $u_k < 2 \Rightarrow u_{k+1} < 2$ appeared in most responses, but the conclusion of the inductive argument was frequently hazy. Thus, for example, many responses showed a validation of H_2, but then went on to claim H_n to be true for all $n \geqslant 1$.

(ii) A significant minority of candidates were unable to obtain a valid expansion of $\sqrt{9-\varepsilon}$ and so could make no progress. Actually a few used $9-\varepsilon = \left(3-\frac{\varepsilon}{6}\right)^2 - \frac{\varepsilon^2}{36} \approx \left(3-\frac{\varepsilon}{6}\right)^2$, or equivalent, in an intelligent way and so obtained the required result immediately. Generally, however, the average standard of responses to this part of the question was well below what had been expected.

7 Write down an expression in terms of z and N for the sum of the series

$$\sum_{n=1}^{N} 2^{-n} z^n.$$ [2]

Use de Moivre's theorem to deduce that

$$\sum_{n=1}^{10} 2^{-n} \sin\left(\tfrac{1}{10}n\pi\right) = \frac{1025\sin\left(\tfrac{1}{10}\pi\right)}{2560 - 2048\cos\left(\tfrac{1}{10}\pi\right)}.$$ [6]

9231_w05_er

Question 7

There were relatively few completely correct responses to this question and this outcome, most of all, was due to technical errors.

Almost all responses showed a correct answer to the first part of this question. Subsequently, the majority produced a broad strategy which was fundamentally correct. In particular, the obtaining of a real form for the denominator of the result for $\sum_{n=1}^{10} 2^{-n} e^{n\pi i/10}$ was usually effected accurately. It was in the final stage, where it is necessary to extract the imaginary part of the numerator, that solutions ran into confusion. Few candidates made obvious simplifications as their working developed, e.g., $z = e^{i\pi/10} \Rightarrow z^{10} = e^{i\pi} = -1$ and so arrived at their destination, if at all, only after a lot of unnecessary labour. In contrast, a small minority of candidates produced impressive working to prove what was required with a remarkable economy of effort.

Answer : $\dfrac{z\left(1-\left(\dfrac{z}{2}\right)^N\right)}{2-z}$

1 Express

$$u_n = \frac{1}{4n^2 - 1}$$

in partial fractions, and hence find $\sum_{n=1}^{N} u_n$ in terms of N. [4]

Deduce that the infinite series $u_1 + u_2 + u_3 + \dots$ is convergent and state the sum to infinity. [2]

9231_s06_er

Question 1

The majority of candidates produced a complete and correct response to this question. Very few failed to establish the resolution

$$u_n = \frac{1}{4n^2 - 1} = \frac{1}{4n-2} - \frac{1}{4n+2},$$

and to go on to apply the difference method to obtain

$$\frac{1}{2} - \frac{1}{4N+2} \quad (*)$$

At this stage, there were some notational confusion in that n and not N was used in the sum function.

For the rest, some candidates got involved in complicated convergence tests such as might be used when S_N is unobtainable in a simple form. Here, it is sufficient to show that (*) implies $\lim_{N\to\infty} S_N = \frac{1}{2}$, for such a result establishes both the convergence of $\sum_{n=1}^{\infty} u_n$ as well as its value.

Answers: $\frac{1}{2} - \frac{1}{4N+2}$; $\frac{1}{2}$.

3 Prove by induction, or otherwise, that

$$23^{2n} + 31^{2n} + 46$$

is divisible by 48, for all integers $n \geqslant 0$. [6]

9231_s06_er

Question 3

The overall impression given by about half of all responses was that of an uninformed methodology. The hope, apparently, was that if enough algebra appeared then the argument would look after itself. However, this was far from being the case.

In the first place, it is helpful to define

$$\phi(n) \equiv 23^{2n} + 31^{2n} + 46 \text{ for all non-negative } n.$$

This simple notational expedient will simplify the argument later on. Thus to begin with the inductive hypothesis H_k can be formulated as:

$$H_k : 48|\phi(k) \text{ some integer } k > 0.$$

After some algebra the result

$$\phi(k + 1) = \phi(k) + 48(11.23^{2k} + 20.31^{2k})$$

is obtained and this shows that $H_k \Rightarrow H_{k+1}$. Completion of the inductive argument then requires the observation that H_0 is true, since $\phi(0) = 1 + 1 + 46 = 48$. However, many arguments were deficient in some or all of the following ways.

- There was no coherent definition of the inductive hypothesis,
- The central algebra only got as far as showing

$$H_k \Rightarrow \phi(k + 1) - \phi(k) = 48(11.23^{2k} + 20.31^{2k}).$$

- The hypothesis H_1 was shown to be correct, but then it was claimed later that $n|\phi(n)$ for all $n > 0$.

3 Verify that if

$$v_n = n(n+1)(n+2)\ \dots\ (n+m),$$

then

$$v_{n+1} - v_n = (m+1)(n+1)(n+2)\ \dots\ (n+m).$$ [2]

Given now that

$$u_n = (n+1)(n+2)\ \dots\ (n+m),$$

find $\sum_{n=1}^{N} u_n$ in terms of m and N. [3]

4 Prove by mathematical induction that, for all positive integers n, $10^{3n} + 13^{n+1}$ is divisible by 7. [5]

9231_w06_er

Question 3

In contrast to the previous question, the working here was generally accurate. Very few candidates failed to make some progress.

Most responses showed about the right amount of detail to establish the first result.

For the rest of the question, it was generally understood that the method of differences based on the first result was involved, so that most candidates obtained $\sum_{n=1}^{N} u_n = \frac{v_{N+1} - v_1}{m+1}$ However, a minority of candidates were unable to translate this expression into a correct result in terms of m and N, such as the one given below.

Answer: $\frac{(N+1)(N+2)...(N+1+m)}{m+1} - m!$

Question 4

The majority of responses showed a statement of, or at least implied, a correct inductive hypothesis. H_n. In contrast, a minority of candidates began by identifying H_n with a statement of the question, so indicating a complete misunderstanding of the principle of mathematical induction. This fundamental error has occurred in responses to questions on induction in previous examinations of this syllabus and comment on it has been made in corresponding reports.

The essence of the proof, which requires showing that $7|(10^{3k} + 13^{k+1}) \Rightarrow 7|(10^{3k+3} + 13^{k+2})$ was established by most candidates, even if they had failed to define H_n. In this respect, one must remark that some of the working at this stage was complicated, to say the least, and it is therefore much to the credit of some candidates that they managed to find their way through some very obscure detail.

Finally, the majority of responses showed a satisfactory conclusion to the induction argument. Very few failed to make clear the range of n for which the divisibility property is valid.

1 Verify that

$$\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1} = \frac{2n+1}{(n^2+1)(n^2+2n+2)}.$$ [1]

Use the method of differences to show that, for all $N \geqslant 1$,

$$\sum_{n=1}^{N} \frac{2n+1}{(n^2+1)(n^2+2n+2)} < \frac{1}{2}.$$ [3]

Write down the value of

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+1)(n^2+2n+2)}.$$ [1]

9231_s07_er

Question 1

There were many complete and accurate answers to this question. Almost all candidates were able to establish the first result

$$\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1} = \frac{2n+1}{(n^2+1)(n^2+2n+2)}.$$

Most were then able to use the method of difference to find

$$S_N = \frac{1}{2} - \frac{1}{(N+1)^2+1}.$$

It was then necessary to explain that since $(N+1)^2$ was positive for $N > 1$, then $\frac{1}{(N+1)^2+1} > 0$ and $S_N < \frac{1}{2}$. Many candidates lost the mark for this piece of work, but, nevertheless, were able to state correctly that S_∞ was $\frac{1}{2}$.

Answer: $\frac{1}{2}$.

2 Express

$$\frac{2n+3}{n(n+1)}$$

in partial fractions and hence use the method of differences to find

$$\sum_{n=1}^{N} \frac{2n+3}{n(n+1)}\left(\frac{1}{3}\right)^{n+1}$$

in terms of N. [4]

Deduce the value of

$$\sum_{n=1}^{\infty} \frac{2n+3}{n(n+1)}\left(\frac{1}{3}\right)^{n+1}.$$ [1]

9231_w07_er

Question 2

There were many complete and accurate answers to this question. Almost all candidates were able to establish the result $\frac{2n+3}{n(n+1)} = \frac{3}{n} - \frac{1}{n+1}$. Many were then able to use the method of differences to show that $\sum_{n=1}^{N} \frac{2n+3}{n(n+1)}\left(\frac{1}{3}\right)^{n+1} = \frac{1}{3} - \frac{1}{(N+1)}\left(\frac{1}{3}\right)^{N+1}$, from which they were able to deduce the sum to infinity correctly.

Answers: $\frac{1}{3} - \frac{1}{3^{N+1}(N+1)}$, $\frac{1}{3}$.

3 Prove by induction that, for all $n \geqslant 1$,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{x^2}) = \mathrm{P}_n(x)\mathrm{e}^{x^2},$$

where $\mathrm{P}_n(x)$ is a polynomial in x of degree n with the coefficient of x^n equal to 2^n. [6]

9231_w07_er

Question 3

This proved to be a most difficult question for a large number of candidates.

Often only one or two marks were gained for stating inductive hypothesis and/or demonstrating that the result was true for $n = 1$.

There seemed to be much confusion in the minds of candidates over what constituted a polynomial in x of degree n, so few were able to show that H_k is true $\Rightarrow$ H_{k+1} is true.

The most concise solutions stated that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}P_k(x)\right) = 2x\mathrm{e}^{x^2}P_k(x) + \mathrm{e}^{x^2}P'(x)$ and explained that the first term was the product of e^{x^2} and a polynomial in x of degree $k + 1$, while the second term was the product of e^{x^2} and a polynomial in x of degree $k - 1$, thus producing $\mathrm{e}^{x^2}P_{k+1}(x)$. Occasionally a candidate, who did get to this stage, did not write a full conclusion and so dropped the final mark.